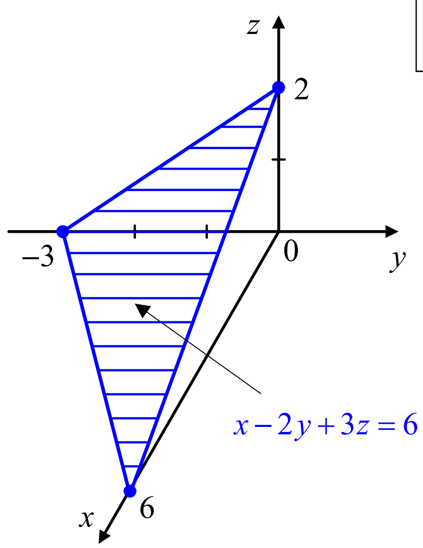
SAMPLES of Solving

1^{st} kind surface integral

TASK: Evaluate the 1^{st} kind surface integral $\iint\limits_S (x+3z)dS$, where **S** is a part of the plane $x-2y+3z=6$, created as a result of intersection of the plane **S** with coordinate planes.

SOLUTION: Since the surface **S** is regular with respect to *z*, it can be given by $z=\varphi(x,y)$, then

$$\iint\limits_S f(x,y,z)\,\boldsymbol{dS} = \iint\limits_{Dxy} f\big(x,y,\varphi(x,y)\big)\cdot\sqrt{1+(z'_x)^2+(z'_y)^2}\cdot dxdy.$$



●) In order to represent the plane **S**: $x-2y+3z=6$, its equation must be rewritten in the form:

$$\frac{x}{6}-\frac{y}{3}+\frac{z}{2}=1.$$

●) In order to evaluate dS, the equation of surface **S** must be rewritten in the form:

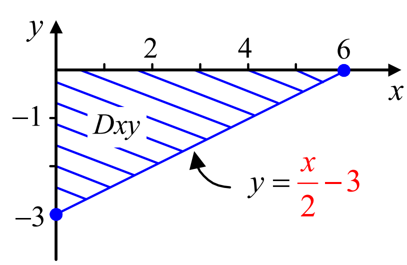
$$z=2-\frac{x}{3}+\frac{2y}{3}, \quad \text{then } z'_x=-\frac{1}{3}, \quad z'_y=\frac{2}{3} \quad \text{un}$$

$$dS=\sqrt{1+\left(z'_x\right)^2+\left(z'\right)_y^2}\,dxdy=\sqrt{1+\frac{1}{9}+\frac{4}{9}}dxdy=\sqrt{\frac{14}{9}}dxdy=\frac{\sqrt{14}}{3}dxdy.$$

●) After substituting, the surface integral is reduced into the double integral:

$$\iint\limits_S (x+3z)dS=\iint\limits_{Dxy}\left(x+3\left(2-\frac{x}{3}+\frac{2y}{3}\right)\right)\cdot\frac{\sqrt{14}}{3}\cdot dxdy=\frac{\sqrt{14}}{3}\iint\limits_{Dxy}(6+2y)\,dxdy=\frac{2\sqrt{14}}{3}\iint\limits_{Dxy}(3+y)\,dxdy=$$

$$=\left[\,Dxy=proj_{xy}S, \quad equation\ of\ the\ line\ is\ \frac{x}{6}-\frac{y}{3}=1, \quad \Rightarrow\ y=\frac{x}{2}-3\,\right]$$



$$=\frac{2\sqrt{14}}{3}\int\limits_0^6 dx\int\limits_{\frac{x}{2}-3}^{0}(3+y)\,dy=$$

$$=\frac{2\sqrt{14}}{3}\int_0^6 dx \int_{\frac{x}{2}-3}^{0}(3+y)\,dy=\frac{2\sqrt{14}}{3}\int_0^6\left(3\left[y\right]\Big|_{\frac{x}{2}-3}^{0}+\left[\frac{y^2}{2}\right]\Big|_{\frac{x}{2}-3}^{0}\right)dx=$$

$$=\frac{2\sqrt{14}}{3}\int_0^6\left(3\left(0-\left(\frac{x}{2}-3\right)\right)+\frac{1}{2}\left(0^2-\left(\frac{x}{2}-3\right)^2\right)\right)dx=\frac{2\sqrt{14}}{3}\int_0^6\left(-\frac{3x}{2}+9-\frac{1}{2}\left(\frac{x^2}{4}-\not{2}\,\frac{x}{\not{2}}3+9\right)\right)dx=$$

$$=\frac{2\sqrt{14}}{3}\int_0^6\left(-\frac{x^2}{8}+\frac{9}{2}\right)dx=\frac{2\sqrt{14}}{3}\left(-\frac{1}{8}\left[\frac{x^3}{3}\right]\Big|_0^6+\frac{9}{2}\left[x\right]\Big|_0^6\right)=\frac{2\sqrt{14}}{3}\left(-\frac{1}{24}\left(6^3-0^3\right)+\frac{9}{2}(6-0)\right)=$$

$$=\frac{2\sqrt{14}}{3}(-9+27)=12\sqrt{14}\,.$$

Answ. The given 1st kind surface integral is equal to $I=12\sqrt{14}$.