Calculeu la integral doble següent

$$\int_0^1 \int_0^{\sqrt{1-x^2}} e^{x^2+y^2}\,\mathrm{d}y\,\mathrm{d}x$$

SOLUCIÓ:

Resoldrem aquesta integral passant a coordenades polars i considerant que el determinant de la matriu jacobiana del canvi de coordenades de cartesianes a polars té com a valor r. Observem-ho,

$$\left.\begin{array}{c} x = r\cos\theta \\ y = r\sin\theta \end{array}\right\} \quad \Rightarrow \quad \left.\begin{array}{c} r = \sqrt{x^2+y^2} \\ \theta = \arctan\frac{y}{x} \end{array}\right\} \quad \Rightarrow$$

$$\Rightarrow \quad J = \begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{pmatrix} = \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix} \quad \Rightarrow \quad |J| = r\cos^2\theta + r\sin^2\theta = r$$

Realitzant doncs aquest canvi de coordenades obtenim la nostra integral reescrita com,

$$\int_0^1 \int_0^{\sqrt{1-x^2}} e^{x^2+y^2}\,\mathrm{d}y\,\mathrm{d}x = \int_0^{\frac{\pi}{2}} \int_0^1 r e^{r^2}\,\mathrm{d}r\,\mathrm{d}\theta = \frac{\pi}{2}\left.\frac{e^{r^2}}{2}\right|_0^1 = \frac{\pi(e-1)}{4}$$